Precalculus

Lesson 14.3: The Tangent Problem: The Derivative

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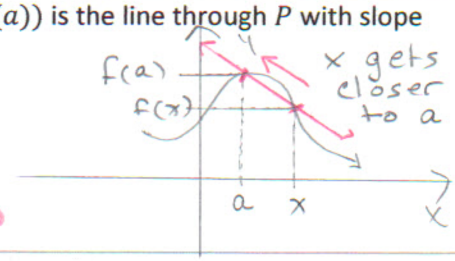
Limits are used to calculate the slope of a line that is tangent to a point on a curved graph. As a little foreshadowing, this tangent line problem and calculating its slope gave rise to the branch of calculus called *differential calculus* which dates back to the early to mid- 1600s!

If we think about our geometry definition of a tangent line, it is words to the effect of: A line that just touches a curve at one point, without cutting across it. In calculus it is a little more than that as stated below:

The tangent line to the curve $y = f(x)$ at the point $P(a, f(a))$ is the line through P with slope

$$m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$



a.k.a. slope of the curve at a point P

Finding a Tangent Line to a Hyperbola

Find an $\boxed{\text{equation}}$ of the tangent line to the hyperbola $y = \frac{3}{x}$ at the point (3,1).

$f(x) = \frac{3}{x}$ $P(a, f(a))$

$a = 3$

$f(a) = 1$

$$m = \lim_{x \to 3} \frac{f(x) - f(3)}{x - 3}$$

$$m = \lim_{x \to 3} \frac{\left(\frac{3}{x} - 1\right)}{(x-3)} \left(\frac{x}{x}\right)$$

$$= \lim_{x \to 3} \frac{3 - x}{x^2 - 3}$$

factor out −1

$$= \lim_{x \to 3} \frac{-1(\cancel{x-3})}{(x+3)(\cancel{x-3})}$$

$$= \lim_{x \to 3} \frac{-1}{x}$$

$$m = -\frac{1}{3}$$

Slope, need equation

Equation passing through (3,1) $m = -\frac{1}{3}$

$$y - 1 \overset{+1}{=} -\frac{1}{3}(x-3)^{+1}$$

$$y = -\frac{1}{3}x + 1 + 1$$

$$y = -\frac{1}{3}x + 2$$

Standard Form: (if needed)

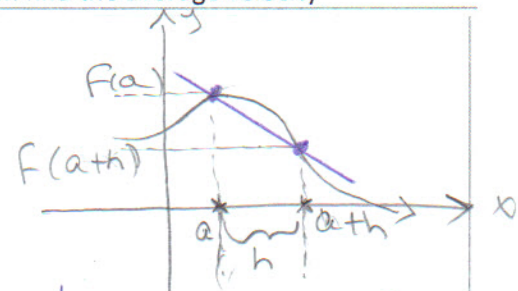
$$3\left(\frac{1}{3}x + y = 2\right)$$

$$x + 3y = 6$$

Whether we find the slope of a tangent line, a velocity or another rate of change, we are basically finding a slope of a line. A common rate of change application problem is velocity. If we have a velocity that is not constant, but changing we can find the average velocity

$$\textit{average velocity} = \frac{\textit{displacement}}{\textit{time}} = \frac{f(a+h) - f(a)}{h}$$

recognize that this is the definiton of $slope = \frac{rise}{run}$



as x gets closer to a, the distance h approaches 0

hence we can use the limit:

$$m = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

Finding a Tangent Line

Find an equation of the tangent line to the curve $y = x^3 - 2x + 3$ at the point (1,2).

$(a, f(a))$

$$m = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h}$$

$$= \lim_{h \to 0} \frac{(1+h)^3 - 2(1+h) + 3 - 2}{h}$$

* Binomial thm or Pascal Δ

$$= \lim_{h \to 0} \frac{(\cancel{1} + \cancel{3h} + 3h^2 + h^3) - \cancel{2} - \cancel{2h} + \cancel{1}}{h}$$

$$= \lim_{h \to 0} \frac{h + 3h^2 + h^3}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{h}(1 + 3h + h^2)}{\cancel{h}}$$

$$m = 1$$

$$\therefore y - 2 = 1(x - 1) \Rightarrow \underline{y = x + 1}$$

Ok, so we are looking at tangent lines and their respective slopes, we also know slope as *rate of change.* This is a very important concept in calculus and applications in science and engineering as many problems deal with motion and hence, the need to be able to determine not only the rate of change of a particle's movement, but the *instantaneous rate of change* of the movement of an object. Because this type of limit is so important it is given a special name and notation:

Definition of a Derivative

The derivative of a function is the slope of the tangent line…so…

rate of change
shows up in many application:
+ rate of a chemical reaction
+ marginal cost in economics

$$f'(a) = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$$

"f prime of a"

Finding a Derivative at a Point

Find the derivative of the function $f(x) = 5x^2 + 3x - 1$ at the number 2.

@ $x = 2 = a$
$y = 20 + 6 - 1$
$y = 25 = f(a)$

$$f'(a) = \lim_{h\to 0}\frac{f(2+h)-f(2)}{h}$$

$$= \lim_{h\to 0}\frac{5(2+h)^2+3(2+h)-1-25}{h}$$

$$= \lim_{h\to 0}\frac{5(4+4h+h^2)+6+3h-26}{h}$$

$$= \lim_{h\to 0}\frac{\cancel{20}+20h+h^2+\cancel{6}+3h-\cancel{26}}{h}$$

$$= \lim_{h\to 0}\frac{23h+h^2}{h}$$

$$= \lim_{h\to 0}\frac{\cancel{h}(23+\overset{\to 0}{h})}{\cancel{h}} = 23$$

Translation:
$f'(a)$ = derivative = slope; so slope of a line through $(2, 25)$ & tangent to $y = 5x^2+3x-1$ is 23.

Let $f(x) = \sqrt{x}$

Find the derivatives: $f'(a), f'(1), f'(4)$, and $f''(9)$

$$f'(a) = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$$

$$= \lim_{h\to 0}\frac{\sqrt{a+h}-\sqrt{a}}{h}\cdot\frac{\sqrt{a+h}+\sqrt{a}}{\sqrt{a+h}+\sqrt{a}}$$ conjugate

$$= \lim_{h\to 0}\frac{\cancel{a}+\cancel{h}-\cancel{a}\ 1}{\cancel{h}(\sqrt{a+h}+\sqrt{a})}$$

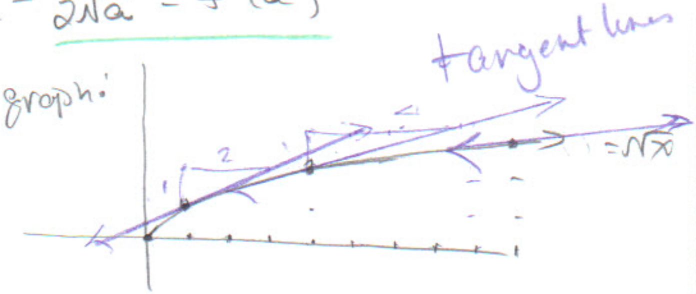
$$= \lim_{h\to 0}\frac{1}{\sqrt{a+\underset{0}{h}}+\sqrt{a}} = \frac{1}{2\sqrt{a}} = f'(a)$$

$$f'(1) = \frac{1}{2\sqrt{1}} = \frac{1}{2}$$

$$f'(4) = \frac{1}{2\sqrt{4}} = \frac{1}{4}$$

$$f'(9) = \frac{1}{2\sqrt{9}} = \frac{1}{6}$$

graph:



The average rate of change of a function f between the numbers a and x as:

$$average\ rate\ of\ change = \frac{change\ in\ y}{change\ in\ x} = \frac{f(x) - f(a)}{x - a}$$

What we want to do is to determine just how fast $f(x)$ is changing at some point $x = a$ This is called the **instantaneous rate of change**.

Instantaneous rate of change of y with respect to x at $x = a$ is the limit of the average rates of change as x approaches a:

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a} = f'(a)$$

Still $\frac{\Delta y}{\Delta x}$

Instantaneous Velocity of a Falling Object: in application problems where $x = t = time\ and\ s = f(t) = displacement\ of\ an\ object$, we have what is known as ***instantaneous velocity.***

If an object is dropped from a height of $3000\ ft$, its distance above the ground (in feet) after **t** seconds is given by $h(t)$. Find the object's instantaneous velocity after 4 seconds.
Modeled by: $h(t) = 3000 - 16t^2$ $\quad h(4) = 3000 - 16(4^2) = 2744\text{ft}$

$$h' = \lim_{t \to 4} \frac{f(t) - f(4)}{t - 4} = \lim_{x \to 4} \frac{3000 - 16t^2 - 2744}{t - 4}$$

$$= \lim_{t \to 4} \frac{256 - 16t^2}{t - 4}$$

$$= \lim_{t \to 4} \frac{-16(t^2 - 16)}{t - 4}$$

$$= \lim_{t \to 4} \frac{-16(t+4)\cancel{(t-4)}}{\cancel{(t-4)}}$$

$$= \lim_{t \to 4} -16(t+4)$$

$$= -16(8) = -128\ \text{ft/sec.}$$