Algebra 2

Lesson 4-8: Cramer's Rule

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Suppose you have a system of equations and you are only interested in the solution of one variable. With the matrix methods learned so far, this is not possible. Cramer's Rule is a theorem in linear algebra, which may be used to solve systems for only the desired variables. This theorem was derived by a Swiss mathematician, Gabriel Cramer, (1704-1752) and uses ratios of determinant values to solve for the variables.

So, given a system of equations: $\begin{cases} 2x + y + z = 3 \\ x - 2y - z = 0 \\ x + 2y + z = 0 \end{cases}$

find (x,y,z)

Convert into a matrix equation: $\begin{bmatrix} 2 & 1 & 1 \\ 1 & -2 & -1 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \mathbf{3} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix}$

We can now make several determinants out of this system. First, **D** is the determinant of the coefficient matrix. We are next going to make three other matrices to help us calculate the values of our variables. This will be done by substituting the one-column constant matrix into the coefficient matrix. We will make three new matrices; one for the x-column values, the y-column values,and the z-column values. $\mathbf{D_x}$ is the determinant formed by replacing the x-coefficient column values with the constant matrix. $\mathbf{D_y}$ is the determinant formed when you replace the y-column values with the constant matrix. Lastly, when you replace the z-column values with the constant matrix you get the $\mathbf{D_z}$ determinant. Look carefully at the matrices. Notice the substitution made to form the matrices:

Constant matrix

$D_x = \begin{vmatrix} \mathbf{3} & 1 & 1 \\ \mathbf{0} & -2 & -1 \\ \mathbf{0} & 2 & 1 \end{vmatrix} =$ _______ $D_y = \begin{vmatrix} 2 & \mathbf{3} & 1 \\ 1 & \mathbf{0} & -1 \\ 1 & \mathbf{0} & 1 \end{vmatrix} =$ _______ $D_z = \begin{vmatrix} 2 & 1 & \mathbf{3} \\ 1 & -2 & \mathbf{0} \\ 1 & 2 & \mathbf{0} \end{vmatrix} =$ _______

$D = \begin{vmatrix} 2 & 1 & 1 \\ 1 & -2 & -1 \\ 1 & 2 & 1 \end{vmatrix} =$ _______

Cramer's rule says that the variable is equal to the ratio of the variable determinant to the coefficient determinant:

$$\boldsymbol{x = \frac{Dx}{D}} \qquad \boldsymbol{y = \frac{Dy}{D}} \qquad \textit{and} \qquad \boldsymbol{z = \frac{Dz}{D}}$$

Summary:

1) *The variable determinants are formed by substituting the constant matrix into the column of the coefficient matrix that houses the coefficients for the given variable.*
2) *The ratio of the variable determinant and the coefficient determinant equals the value of the variab*le.
   $x = \frac{Dx}{D}$, $y = \frac{Dy}{D}$ and $z = \frac{Dz}{D}$
3) Now to find for example the x value, all we need to do is solve the determinants D and D_x.
4) We need to remember with a 3x3 matrix the determinant is solved by expanding the matrix by rewriting column 1 and 2, summing the products of the down-to-the-rights and subtracting the sum of the down-to-the-left products.

Example: Solve using Cramer's Rule –

$\begin{cases} 2x + y = 8 \\ x - y = -2 \end{cases}$	
$\begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix} + \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \mathbf{8} \\ \mathbf{-2} \end{bmatrix}$	1. Set up matrix equation
$D = \begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix} = -3$ $\quad D_x = \begin{vmatrix} 8 & 1 \\ \mathbf{-2} & -1 \end{vmatrix} = -6$	2. Set up determinants and solve
$D_y = \begin{vmatrix} 2 & \mathbf{8} \\ 1 & \mathbf{-2} \end{vmatrix} = -12$	3. Solve the ratios of D_x /D and D_y/D .
$x = \frac{-6}{-3} = 2$ $\quad y = \frac{-12}{-3} = 4$	4. Check your answers for (x,y)